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Route to non-Gaussian statistics in convective turbulence

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Motived by the work of Li and Meneveau [Phys. Rev. Lett. **95**, 164502 (2005)], we propose and solve a model for the Lagrangian evolution of both longitudinal and transverse velocity and temperature increments for Boussinesq convection. From this model, the short-time evolution of an initially imposed Gaussian joint probability density function (PDF) of both velocity and temperature increments is computed analytically and the trend to non-Gaussian statistics shown in a quantitative way. Predictions for moments of the joint PDF are obtained and their behavior analyzed with respect to known experimental and numerical results. The obtained results do not depend on the model free parameters, a fact in favor of their robustness.

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Non-Gaussian fluctuations are ubiquitous features in turbulence, ranging from astrophysical plasma turbulence [1] to the (apparently) simpler world of passive scalar turbulence [2]. The key observation is that the probability density functions of spatial velocity increments across small distances shows considerably longer tails than the ones of a Gaussian distribution. This phenomenon is directly related to the intermittency and anomalous scaling of equal-time structure functions [3].

A fully consistent theoretical description of the mechanisms at the origin of intermittency has been provided in the last decade for passive scalar advection (see [4] for a review) by self-similar Gaussian white-in-time velocity fields [5]. The same theory does not apply to the case of general advection. Nevertheless, strong numerical evidence [6] supports the idea that the same mechanism at the origin of intermittency also operates in the latter case. A fundamental link between geometry and intermittency arises from [6] which has also been detected in Navier-Stokes turbulence [7].

Although intermittency usually emerges in systems displaying cascadelike mechanisms for the energy transfer, a simple Lagrangian model that does not invoke energy cascade and nevertheless shows anomalous corrections to the dimensional predictions of Kolmogorov type has recently been presented by Chevillard and Meneveau in Ref. [8]. In this model, intermittency corrections to the dimensional predictions are induced by the mechanisms at the origin of non-Gaussian statistics identified by Li and Meneveau [9,10] for a simpler Lagrangian model. The latter model was derived from the Navier-Stokes equations coupled to the passive scalar equation and consists of a nonlinear dynamical system for the Lagrangian evolution of increments both of the passive scalar and of the longitudinal and transverse velocity. The integration of the model equations allowed the authors to numerically reconstruct the time evolution of an initially imposed joint probability density function (PDF) for velocity and scalar increments and to capture the basic mechanisms at the origin of non-Gaussian statistics.

In this Rapid Communication we show that Li and Meneveau's idea can be fruitfully carried over to quantitatively describe and predict known features of turbulent convection, including the lack of Gaussianity for the statistics of velocity and temperature increments. The obtained model equations can be solved analytically and the expression for the time-dependent joint PDFs of both velocity and temperature increments at short time extracted. The regime analyzed by Li and Meneveau is obtained in our model as a particular case. Moreover, the fully analytical formulation allows us to identify an inconsistency in the models [9,10]; we propose and apply a strategy to overcome it.

The starting point of our analysis is the three-dimensional incompressible Boussinesq convection problem ruled by the coupled equations

$$\partial_t u_i + \mathbf{u}\cdot\boldsymbol{\partial} u_i = -\frac{1}{\rho}\partial_i p + [1-\beta(T-T_0)]g_i + \nu\partial^2 u_i,$$

$$\partial_t T + \mathbf{u}\cdot\boldsymbol{\partial} T = \kappa\partial^2 T,$$

where T is the temperature field (with T_0 a reference value), $\mathbf{g}=-g\hat{\mathbf{z}}$ is the gravitational acceleration, β is the thermal expansion coefficient, and κ and ν are the molecular diffusivity and viscosity, respectively.

Our aim here is to find equations for the velocity and temperature increments between points separated by a distance, say l. When the scale l belongs to the interval of scales where both velocity and temperature are smooth, the increments of the latter two fields over the distance l are determined by knowledge of velocity and temperature gradients via a simple Taylor expansion. A natural way to define a scale l at which dynamical fields are smooth is to act with a filter on the governing equations. In doing so, taking the gradient of the resulting equations for the filtered fields $\bar{\mathbf{u}}$ and $\bar{T}$, it is not difficult to obtain (see [11,12] for the particular case without buoyancy) the following equations for $\bar{A}_{ij} \equiv \partial_i \bar{u}_j$ and $\bar{B}_i \equiv \partial_i \bar{T}$:

$$\dot{\bar{A}}_{ij} = -\bar{A}_{ik}\bar{A}_{kj} + \beta g \bar{B}_i \delta_{jd} + \frac{D}{d}\delta_{ij} + H_{ij} \tag{1a}$$

$$\dot{\bar{B}}_i = -\bar{A}_{ij}\bar{B}_j + K_i. \tag{1b}$$

The time derivatives here are Lagrangian derivatives (i.e., the rate of change along the local smoothed field),

$D\equiv\bar{A}_{ij}\bar{A}_{ji}-\beta g\bar{B}_d$, $H_{ij}\equiv-[\partial^2_{ij}\bar{p}-(1/d)\delta_{ij}\partial^2\bar{p}]-[\partial^2_{ik}\tau^u_{jk}-(1/d)\delta_{ij}\partial^2_{lk}\tau^u_{lk}]+\nu\partial^2\bar{A}_{ij}$, $K_i\equiv-\partial^2_{ij}\tau^T_j+\kappa\partial^2\bar{B}_i$, τ^u_{ij} and τ^T_i being the usual subgrid scale stresses for velocity and temperature, respectively, and d is the space dimension (from now on $d=2,3$).

Let us now consider velocity and temperature differences over a distance r close to l. From the smoothness of the two fields we have

$$\delta\bar{u}_i(\mathbf{r},t)\equiv\bar{u}_i(\mathbf{x}+\mathbf{r})-\bar{u}_i(\mathbf{x})\sim\bar{A}_{ji}r_j, \tag{2a}$$

$$\delta\bar{T}(\mathbf{r},t)\equiv\bar{T}(\mathbf{x}+\mathbf{r})-\bar{T}(\mathbf{x})\sim\bar{B}_ir_i. \tag{2b}$$

Under the assumption of homogeneous turbulent fluctuations, $\bar{A}_{ij}$ and $\bar{B}_i$ do not depend on the position $\mathbf{x}$ and, moreover, $\langle\delta\bar{u}_i(\mathbf{r},t)\rangle=\langle\delta\bar{T}(\mathbf{r},t)\rangle=0$, where the angular brackets denote space or time averages. Expressions for the transverse and longitudinal velocity increments and temperature differences over the fixed distance l immediately follow from (2) and the smoothness of the velocity field:

$$\delta u=l\bar{A}_{ij}\hat{r}_i\hat{r}_j, \tag{3a}$$

$$\delta v=l|P_{ij}\bar{A}_{kj}\hat{r}_k|=l\bar{A}_{ij}\hat{r}_i\hat{e}_j, \tag{3b}$$

$$\delta T=l\bar{B}_i\hat{r}_i. \tag{3c}$$

Here $\hat{\mathbf{r}}$ and $\hat{\mathbf{e}}$ are the unit vectors in the directions of the longitudinal and the transverse velocity components and $P_{ij}\equiv\delta_{ij}-\hat{r}_i\hat{r}_j$ is the usual transverse projector.

In order to obtain equations for δu, δv, θ (i.e., the angle between $\hat{\mathbf{e}}$ and $\hat{\mathbf{e}}_c$, the unit vector orthogonal to $\hat{\mathbf{r}}$ and lying on the same plane of $\hat{\mathbf{z}}$), δT, and λ (i.e., the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$), the first step is to take the material derivatives of (3). An explicit dependence on $\dot{\bar{A}}_{ij}$, $\dot{\bar{B}}_i$, and $\dot{r}_k$ emerges as a consequence of time derivatives. The latter quantities can immediately be obtained exploiting Eqs. (1) and recalling that we are dealing with a smooth flow on l, for which $\dot{r}_i=\bar{A}_{ji}r_j$. The resulting equations in d dimensions (here $d=2,3$) read

$$\dot{\delta u}=\frac{1}{l}\frac{2-d}{d}\delta u^2+\frac{1}{l}\delta v^2+\beta g\,\delta T\cos\lambda+lH'_{rr}, \tag{4a}$$

$$\dot{\delta v}=-\frac{2}{l}\delta u\,\delta v+\beta g\,\delta T\sin\lambda\cos\theta+lH_{re}, \tag{4b}$$

$$\dot{\theta}=\frac{1}{l}\delta v\frac{\sin\theta}{\tan\lambda}-\beta g\frac{\delta T}{\delta v}\sin\lambda\sin\theta+l\frac{H_{rn}}{\delta v}, \tag{4c}$$

$$\dot{\delta T}=-\frac{1}{l}\delta u\,\delta T+lK_r, \tag{4d}$$

$$\dot{\lambda}=-\frac{1}{l}\delta v\cos\theta, \tag{4e}$$

where the subscripts r, e, and n label the projections onto the three unit vectors of the rotating orthonormal basis, $\hat{\mathbf{r}}$, $\hat{\mathbf{e}}$, and $\hat{\mathbf{n}}=\hat{\mathbf{r}}\times\hat{\mathbf{e}}$, respectively (e.g., $H_{re}=H_{ij}\hat{r}_i\hat{e}_j$). In Eq. (4a) the imposition of the divergenceless condition for the velocity field [13] allows one to extract from D the term $2\delta u^2/l^2$. In plain words, $D=2\delta u^2/l^2+D'$, D' being not closed with respect to the Lagrangian variables of system (4). The function $H'_{rr}=H_{rr}+D'/d$ thus appears in (4a). The two-dimensional case $d=2$ corresponds to $\theta\equiv0$ with $H_{rn}\equiv0$, which identically satisfies Eq. (4c).

Notice that, when δT is a passive scalar and δu is a white-in-time random process, neglecting K_r, a non-Gaussian statistics for δT is expected by virtue of the arc sine law [14]: keeping the same sign for the whole walk and equipartition of the time between positive and negative values are indeed the most and the least probable events, respectively.

Coming back to the system (4), the functions $H'_{rr}(t)$, $H_{re}(t)$, $H_{rn}(t)$, and $K_r(t)$ are unknown and give rise to the usual closure problem typical of nonlinear systems. In Ref. [10] the closure problem has been tackled by assuming $H'_{rr}=H_{re}=H_{rn}=K_r=0$. We will see later that such an assumption actually turns out to be incompatible with homogeneity. To overcome the problem, we take a different point of view with respect to Refs. [9,10] by making explicit the time dependency of $H'_{rr}(t)$, $H_{re}(t)$, $H_{rn}(t)$, and $K_r(t)$. In view of the fact that we will focus our attention on the short-time evolution of the system (4) we can perform a Taylor expansion for the four above unknown functions up to $O(t^4)$ in the present study:

$$H'_{rr}(t)=\sum_{i=0}^{3}H'^{(i)}_{rr}t^i+O(t^4), \tag{5a}$$

$$H_{re}(t)=\sum_{i=0}^{3}H^{(i)}_{re}t^i+O(t^4), \tag{5b}$$

$$H_{rn}(t)=\sum_{i=0}^{3}H^{(i)}_{rn}t^i+O(t^4), \tag{5c}$$

$$K_r(t)=\sum_{i=0}^{3}K^{(i)}_{r}t^i+O(t^4), \tag{5d}$$

with coefficients $H'^{(i)}_{rr}$, $H^{(i)}_{re}$, $H^{(i)}_{rn}$, and $K^{(i)}_r$ to be determined. As we will see, eight of 16 of these coefficients remain free, the others being fixed by the imposition of $\langle\delta u\rangle=\langle\delta T\rangle=0$ at the $O(t^4)$.

Unlike what happens in the homogeneous and isotropic limit of purely hydrodynamic turbulence of Refs. [9,10], here the angular dependences must explicitly be taken into account as a consequence of the intimate anisotropy of our system. By its very definition, the model system (4) is justified only for short-time evolutions. It is thus meaningful to solve it for short times by means of low-order Taylor expansions. Here we consider the solution up to order t^4. The latter will be enough to draw our main conclusions. The resulting expressions for $\delta u(t)$, $\delta v(t)$, $\theta(t)$, $\delta T(t)$, and $\lambda(t)$ correct up to $O(t^4)$ turn out to be quite long and not particularly informative. For this reason they will not be reported here.

Once we have extracted the initial conditions as a function of the solutions at time t, we can easily determine the time evolution of an initially imposed joint PDF and, consequently, of any its moment averages being on the evolved joint PDF. To be more specific, let $\mathcal{P}_0 \equiv \mathcal{P}(\delta u_0, \delta v_0, \theta_0, \delta T_0, \lambda_0; 0)$ be the joint PDF at the initial time $t=0$. The time evolution of $\mathcal{P}$ is simply obtained from the relationship

$$\mathcal{P}(\delta u, \delta v, \theta, \delta T, \lambda; t) d\delta u\, d\delta v\, d\theta\, d\delta T\, d\lambda = \mathcal{P}_0 d\delta u_0 d\delta v_0 d\theta_0 d\delta T_0 d\lambda_0.$$

An initially imposed joint PDF thus evolves according to the evolution equation $\mathcal{P}(\delta u, \delta v, \theta, \delta T, \lambda; t) = J_{t0}\mathcal{P}_0$, J_{t0} being the Jacobian of the inverse transformation between times t and 0. In more detail, assuming for $\mathcal{P}_0$ a given form (e.g., a Gaussian form for velocity and temperature and a solid angular uniform distribution, as we will see later), the calculation of the Jacobian at the order t^4 yields the expression for the $O(t^4)$-evolved PDF. This expression clearly contains all relevant statistical information about the short-time evolution of any two-point statistical indicator. More precisely, to calculate the ensemble average $\langle \mathcal{O} \rangle = \int \mathcal{O}\mathcal{P}\, d\delta u\, d\delta v\, d\theta\, d\delta T\, d\lambda$ of some quantity $\mathcal{O}$ at time t, we simply have used the relation

$$\langle \mathcal{O}(\delta u, \delta v, \theta, \delta T, \lambda; t) \rangle = \langle \mathcal{O}(\delta u(\delta u_0, \ldots, \lambda_0), \delta v(\delta u_0, \ldots, \lambda_0), \theta(\delta u_0, \ldots, \lambda_0), \delta T(\delta u_0, \ldots, \lambda_0), \lambda(\delta u_0, \ldots, \lambda_0); t) \rangle_0, \tag{6}$$

where $\langle \mathcal{O} \rangle_0 = \int \mathcal{O}\mathcal{P}_0 d\delta u_0 d\delta v_0 d\theta_0 d\delta T_0 d\lambda_0$. The advantage of expression (6) is that it does not involve the evolved PDF, thus avoiding the calculation of the Jacobian J_{t0}. Let us extract therefore some standard observables and show that they behave consistently with known results of turbulent convection.

At first we have to assume an initial joint PDF. Assuming at $t=0$ homogeneous and isotropic conditions and Gaussianity for the three components of the velocity increment—hence that δv_0 is distributed according to the Abel transform of the transverse Gaussian (see [15])—in the three-dimensional case we have

$$\mathcal{P}_0(d=3) = \frac{e^{-\delta u_0^2/2\sigma_u^2}}{\sqrt{2\pi}\sigma_u} \frac{\delta v_0 e^{-\delta v_0^2/\sigma_v^2}}{\sigma_v^2} \frac{1}{2\pi} \frac{e^{-\delta T_0^2/2\sigma_T^2}}{\sqrt{2\pi}\sigma_T} \frac{\sin\lambda_0}{2},$$

while in the two-dimensional one

$$\mathcal{P}_0(d=2) = \frac{e^{-\delta u_0^2/2\sigma_u^2}}{\sqrt{2\pi}\sigma_u} \frac{e^{-\delta v_0^2/2\sigma_v^2}}{\sqrt{2\pi}\sigma_v} \frac{e^{-\delta T_0^2/2\sigma_T^2}}{\sqrt{2\pi}\sigma_T} \frac{1}{2\pi}$$

[but with different ranges in the latter case: $dv_0 \in (-\infty, +\infty)$ and $\lambda \in [0, 2\pi)$].

A second point to address is relative to the constraint arising from the requirement of dealing with homogeneous fluctuations. In the three-dimensional case (see [16] for the two-dimensional case), assuming $\sigma_u = \sigma_v$, the imposition $\langle \delta u \rangle = \langle \delta T \rangle = 0$ at $O(t^4)$ leads to $K_r^{(i)} = 0$, $i=0,\ldots,3$, and $H_{rr}'^{(0)} = -(5/3)\sigma_u^2/l^2$ (see [17]). The remaining parameters of $H_{rr}'(t)$ can be arranged as functions of $H_{re}^{(0)}$, $H_{re}^{(1)}$, and $H_{re}^{(2)}$. For the sake of example, after simple but lengthy algebra one obtains $H_{rr}'^{(1)} = -(\sqrt{2\pi}\sigma_u/l)H_{re}^{(0)}$. The explicit expressions for $H_{rr}'^{(2)}$ and $H_{rr}'^{(3)}$ are quite long and not particularly informative; they are, however, reported in [18]. The values of $H_{re}^{(i)}$ and of $H_{rn}^{(i)}$ remain undetermined for $i=0,\ldots,3$. The important point we anticipate here is that the results we are going to show do not depend on these free parameters. In view of the fact that we do not have to fix any other free parameter, the conclusions we will draw in the following do appear very robust and independent from the model details.

Once the homogeneity condition is met, the first point we want to assess is the sign of both energy and temperature variance fluxes. To do that it is enough to study the sign of $S_u = \langle (\delta u)^3 \rangle / \langle (\delta u)^2 \rangle^{3/2}$ and $\Phi_T = \langle (\delta T)^2 \delta u \rangle$. The negative (positive) sign means a flux toward small (large) scales. At $O(t^2)$ we found

$$S_u = -\frac{6\sigma_u(d-2)}{ld}t,$$

$$\Phi_T = -\frac{2\sigma_u^2\sigma_T^2}{l}t.$$

It is evident from the above expressions that, while Φ_T maintains the same (negative) sign for $d=2$ and $d=3$, this is not the case for the skewness factor, which becomes zero on passing from $d=3$ to $d=2$. This result is a reminiscence of the fact that temperature variance flows toward small scales for both $d=2$ and $d=3$, while this does not happen for the energy flux, which stops flowing to small scales on passing from $d=3$ to $d=2$ (see [19]).

Another interesting point to investigate is whether the initially imposed Gaussian fluctuations deform as time runs to generate a non-Gaussian PDF. The easiest way to see that is to calculate the flatness indicator for both velocity and temperature fluctuations: $F_u \equiv \langle (\delta u)^4 \rangle / \langle (\delta u)^2 \rangle^2$ and $F_T \equiv \langle (\delta T)^4 \rangle / \langle (\delta T)^2 \rangle^2$. Up to $O(t^3)$ their expressions read

$$F_u = 3 + \frac{72\sigma_u^2(d-2)^2}{l^2 d^2}t^2, \tag{7a}$$

$$F_T = 3 + \frac{12\sigma_u^2}{l^2}t^2. \tag{7b}$$

It is worth observing from (7) that at $O(t^2)$ buoyancy does not enter into play. At that order, temperature and velocity fluctuations evolve as if the former field were a passive scalar. In this case our results can be thus compared with those of [10] where the passive scalar case has been numerically analyzed.

$F_u = F_T = 3$ would correspond to the Gaussian case. In both cases the second-order corrections $F_u^{(2)}$ and $F_T^{(2)}$ are positive: this is the fingerprint of the emergence of non-Gaussian statistics. In agreement with [10] for $d=2$ we found $F_u=3$ and $F_T > 3$. This is consistent with the fact that the degree of non-Gaussianity in the two-dimensional inverse energy cascade appears to be very small [19,20] and nevertheless intermittency is generated by the dynamics ruling temperature fluctuations [19,21]. Again in agreement with the numerical results of Ref. [10] we found $F_T > F_u$. This result is consistent with the experimental results of Ref. [22] where passive scalar statistics is shown to be more intermittent than the advective velocity statistics.

Buoyancy contributes to the flatness factors at the $O(t^4)$, and its correction to these factors relative to the purely hydrodynamic case is given by

$$F_u(\beta g \neq 0) - F_u(\beta g = 0) = \left(\frac{\beta g \sigma_T}{l}\right)^2 \left[a_d + b_d\left(\frac{l\beta g \sigma_T}{\sigma_u^2}\right)^2\right] t^4, \tag{8a}$$

$$F_T(\beta g \neq 0) - F_T(\beta g = 0) = \left(\frac{\beta g \sigma_T}{l}\right)^2 c_d t^4, \tag{8b}$$

where a_d, b_d, and c_d are positive numbers depending on the dimension d: $a_2=7/2$, $b_2=3/8$, $c_2=19/2$, $a_3=155/27$, $b_3=4/15$, and $c_3=50/9$. Since buoyancy contributions are positive, (8) tell us that generation of non-Gaussianity is more effective in convective environments than in situations where the buoyancy is absent. This is consistent with the fact that velocity fluctuations in turbulent convection are more intermittent than fluctuations in turbulent flows where buoyancy does not drive the dynamics [23].

In conclusion, a model for the short-time evolution of an initially imposed fluctuation in convective turbulence has been proposed and investigated in analytical terms. Remarkably, well-known features of fully developed turbulent convection are already present at the very initial stage of the temporal evolution predicted by the model, whose results do not show an explicit dependence on the model free parameters. Interesting aspects left to future research are on how to fix the remaining model free parameters with the aim of understanding their possible role to reach a statistically steady state and on the extension of the present analysis to other turbulent systems including magnetohydrodynamics and non-Newtonian turbulence.

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[13] In our simplified model this is the only way that flow incompressibility is accounted for. This is clearly an approximated way by virtue of the fact that the imposition $\bar{A}_{ii}=0$ yields a relation not closed in our Lagrangian fields and thus not directly usable.
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[15] Let θ be uniformly distributed in $[0,2\pi)$ and δv denote the magnitude of the transverse-velocity increment vector. Defining $\delta v_c \equiv \delta v \cos\theta$ and assuming for the latter a Gaussian distribution, then the PDF of δv is the Abel transform of the Gaussian.
[16] For $d=2$ all functions (5) are determined: $H'^{(1)}_{rr}=H'^{(3)}_{rr}=H^{(i)}_{re}=H^{(i)}_{rn}=K^{(i)}_{r}=0$, $i=0,\ldots,3$; $H'^{(0)}_{rr}=-\sigma_u^2/l^2$ and $H'^{(2)}_{rr}=-4\sigma_u^4/l^4-1/2\beta^2 g^2\sigma_T^2/l^2$ are fixed at constant negative values. More details can be found at http://www.ge.infn.it/~tizzi/worksheet_2d.html
[17] Let us show that $H'^{(0)}_{rr}=-(5/3)\sigma_u^2/l^2$. From (4a), Taylor expanding its left-hand side and averaging, one has $\langle\delta u\rangle=[-\langle\delta u_0^2\rangle/(3l)+\langle\delta v_0^2\rangle/l+lH'^{(0)}_{rr}]t+O(t^2)$, where averages are at $t=0$ and, consequently, $\langle\delta T_0\cos\lambda_0\rangle=0$. In order to have $\langle\delta u\rangle=0$ at the lowest order one necessarily gets $H'^{(0)}_{rr}=-(5/3)\sigma_u^2/l^2$. Similar simple algebra leads to the expressions for the other H'_{rr} and K_r components.
[18] The expressions of both δu up to $O(t^4)$ and H'_{rr} up to $O(t^3)$ are very long and can be found at http://www.ge.infn.it/~tizzi/worksheet_3d.html.
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